

# Liquid-gas and other unusual thermal phase transitions in some large- $N$ magnets

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## Abstract

Much insight into the low temperature properties of quantum magnets has been gained by generalizing them to symmetry groups of order  $N$ , and then studying the large  $N$  limit. In this paper we consider an unusual aspect of their finite temperature behavior—their exhibiting a phase transition between a perfectly paramagnetic state and a paramagnetic state with a finite correlation length at  $N = \infty$ . We analyze this phenomenon in some detail in the large “spin” (classical) limit of the  $SU(N)$  ferromagnet which is also a lattice discretization of the  $CP^{N-1}$  model. We show that at  $N = \infty$  the order of the transition is governed by lattice connectivity. At finite values of  $N$ , the transition goes away in one or less dimension but survives on many lattices in two dimensions and higher, for sufficiently large  $N$ . The latter conclusion contradicts a recent conjecture of Sokal and Starinets [5], yet is consistent with the known finite temperature behavior of the  $SU(2)$  case. We also report closely related first order paramagnet-ferromagnet transitions at large  $N$  and shed light on a violation of Elitzur’s theorem at infinite  $N$  via the large  $q$  limit of the  $q$  state Potts model, reformulated as an Ising gauge theory.

*Key words:*  $1/N$  expansion, quantum magnetism, nonlinear  $\sigma$  model,  $CP^{N-1}$  model, phase transitions

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## 1 Introduction

The properties of quantum antiferromagnets in low dimensions have been intensely studied over the past decade and a half. Much insight has been gained by large  $N$  treatments based on generalizing the symmetry group from  $SU(2)$  to either  $SU(N)$  or  $Sp(N)$ , especially in two dimensions where exact

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solutions are not available. These reformulations involve the representation of the spins by bilinears in fermionic or bosonic “spinon” operators at the cost of introducing local constraints on their number and an associated gauge invariance [2–4]. At  $N = \infty$  the constraints are trivially solved and a purely quadratic problem results. Most of this effort has gone into elucidating the zero temperature phase diagram and has also involved going beyond the  $N = \infty$  limit by thinking about the structure of the gauge theory that results if the spinons are integrated out.

An oddity from the early work using bosons is the report [3], on a square lattice, of a finite-temperature *phase* at  $N = \infty$  with no intersite correlations—a perfect paramagnet. At low temperatures the energy cost compels spins to align (in ferromagnets) or antialign (in antiferromagnets). In the language of Schwinger bosons, this is seen as a correlation or anticorrelation of boson flavors on adjacent sites. At higher temperatures, free energy  $F = E - TS$  is dominated by entropy. When the number of boson flavors  $N \rightarrow \infty$  (with the coupling constant appropriately rescaled  $J \mapsto J/N$ ), the entropy of the disordered state completely overpowers the energy cost, so that neighboring sites become *perfectly* uncorrelated above a certain temperature. (At large but finite  $N$  the high-temperature phase has nonvanishing correlations.) Such a phase must then be separated from the lower temperature paramagnetic state with finite intersite correlations by a phase transition. This phase transition clearly has no analog in the physical  $SU(2)$  problem and is therefore an embarrassment for the large- $N$  approach.<sup>1</sup> We have found that an essentially identical transition occurs in the  $SU(N)$  generalization of the Heisenberg ferromagnet at  $N = \infty$ . Understanding its fate at finite  $N$  is equally a matter of interest.

In this paper we investigate this transition in some detail. To make life simpler we have restricted ourselves to the  $SU(N)$  ferromagnet, although much of what we say should apply *mutatis mutandis* to the  $Sp(N)$  antiferromagnet. We make one further simplification, that of taking the large “spin” or boson density limit at any fixed  $N$ , which renders the problem classical without destroying the transition of interest.<sup>2</sup> For the classical  $SU(N)$  ferromagnet, we first examine the  $N = \infty$  solution carefully and show that the transition is first order on the square lattice, a fact which will be crucial in the following. We analyze a number of other lattices and find that this is the outcome on all lattices that have a shortest closed loop of length three or four. On other lattices, such as the honeycomb or the linear chain, which lack such loops,

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<sup>1</sup> A similar embarrassment arises in the infinite- $N$  solution of the Kondo problem which exhibits truly non-analytic behavior at a finite temperature. There the exact Bethe Ansatz solutions were used to show that this happens only at infinite  $N$ ; otherwise it is a crossover that sharpens continuously as  $N$  is increased [6,7].

<sup>2</sup> Much of what we have to say should go through at large but not infinite “spin”—at finite temperatures this distinction is quantitative.

the transition is continuous. The order of the transition at  $N = \infty$  is thus influenced by lattice connectivity and not dimensionality. However, life does become more interesting in  $d > 2$  where ferromagnetic states that break the  $SU(N)$  symmetry become stable to the Mermin-Wagner fluctuations contained in the  $N = \infty$  theory. Now it is possible for such states to “piggy-back” on the finitely correlated paramagnets and first order transitions between the perfect paramagnet and ferromagnetic states result in cases where the “underlying transition” is predicted to be first order “enough”. The corresponding transition in the  $Sp(N)$  antiferromagnet has been found recently by DeSilva and co-workers [8].

We turn next to the survival of this transition at finite values of  $N$ . We offer strong evidence for the following conclusions.

- (a) In  $d = 1$  it goes away, as it must on general grounds.
- (b) In  $d = 2$  the first order transitions survive for sufficiently large values of  $N$  but terminate at an Ising critical endpoint; the critical value  $N_c$  is likely too large to be seen in feasible simulations.
- (c) In  $d = 3$  and above the first order transitions again survive. In cases where they are preempted by first order transitions to the ferromagnetic state at  $N = \infty$  the latter transition again survives at large  $N$  and presumably turns continuous before the  $SU(2)$  limit is reached.
- (d) In cases where the transition is continuous at  $N = \infty$ , we conclude that the transition goes away at finite  $N$ .

The classical  $SU(N)$  model we study has a pre-history for it is a lattice version of the  $CP^{N-1}$  model. Previous workers, most notably Sokal and Starinets [5], have concluded that the transition in the lattice model is an artefact of  $N = \infty$  in *all* dimensions. Their arguments are based on an exact solution of the  $d = 1$  problem, a violation of gauge invariance at  $N = \infty$  and the apparent lack of such a transition at small  $N$  in simulations. In a companion paper to ours, Fendley and one of us (OT) have independently solved the  $d = 1$  case and so we are in agreement with Sokal and Starinets on that. We will argue below that the breakdown of gauge invariance is misleading and in the course of this argument we will appeal to a similar breakdown in an Ising gauge reformulation of the  $q$  state Potts model that exhibits a phase transition with a family resemblance to the ones at issue and where one can appeal to well-established results. We will argue that the transition at issue has the character of a liquid-gas transition in that the two phases can be smoothly continued into one another. The two paramagnetic phases have different energy densities; the difference varies with  $N$  and vanishes at some critical value  $N_c$ . Finally, we will compute the dimensionless surface tension at coexistence for the infinite- $N$  problem and show that it is small and thereby conclude that the transition terminates at rather large values of  $N$ , consistent with the failure to observe it in simulations.

In the balance of this section we introduce the quantum and classical Hamiltonians of interest. In Section 2 we carry out a saddle point analysis at  $N = \infty$ . In Section 3 we consider the finite (but large)  $N$  problem. We summarize our conclusions in Section 4.

### 1.1 Hamiltonians

The quantum problems we have in mind are the bosonic  $SU(N)$  generalization of the Heisenberg antiferromagnet:

$$H = -\frac{J}{2N} \sum_{\langle ij \rangle} (b_{i\alpha}^\dagger b_{j\alpha})(b_{j\beta}^\dagger b_{i\beta}) \quad (1)$$

and the bosonic  $Sp(N)$  generalization of the Heisenberg antiferromagnet:

$$H = -\frac{J}{2N} \sum_{\langle ij \rangle} (\mathcal{J}^{\alpha\beta} b_{i\alpha}^\dagger b_{j\beta}^\dagger)(\mathcal{J}_{\gamma\delta} b_{i\gamma} b_{j\delta}). \quad (2)$$

Here  $\mathcal{J}$  is an antisymmetric block-diagonal matrix  $2N \times 2N$ :

$$\mathcal{J}^{\alpha\beta} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}. \quad (3)$$

The Greek indices run from 1 to  $N$  ( $2N$  for the antiferromagnet) and  $\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha} = n_b$  fixes the “spin”. For  $SU(2) \equiv Sp(1)$ ,  $n_b = 2S$  does fix the spin. Note that these generalizations, obtained by replacing the spin operators by their bosonic “square roots” come with local constraints and hence a local gauge invariance which we make more explicit below.

Considerable insight has been gained from considering the mean field theory that results in the  $N \rightarrow \infty$  limit taken while keeping  $\kappa = n_b/N$  fixed. However this limit also exhibits a finite temperature phase transition between two paramagnetic phases, first discussed for the  $SU(N)$  case by Arovas and Auerbach [3].

To study this transition in more detail, we will study an easier limit, that of  $\kappa \rightarrow \infty$  at finite temperatures. With appropriately rescaled variables and

units such that

$$b_{i\alpha} = z_{i\alpha} n_b / N, \quad J(n_b / N)^2 = 2,$$

this yields a classical partition function governed, in the  $SU(N)$  case, by the classical energy

$$E = -\frac{1}{N} \sum_{\langle ij \rangle} |z_{i\alpha}^* z_{j\alpha}|^2, \quad (4)$$

defined in terms of a complex  $N$ -vector  $z$  obeying  $z^\dagger \cdot z = N$ . This Hamiltonian is invariant under global  $SU(N)$  rotations, and also under the aforementioned local gauge transformations  $z_i \rightarrow e^{i\alpha_i} z_i$  at any site  $i$ . The vector  $z$  takes values on a complex sphere  $U(N)/U(N-1)$ , which as a manifold is identical to the real  $2N-1$ -sphere  $O(2N)/O(2N-1)$ . However, the  $U(1)$  gauge symmetry can be used to effectively reduce the number of degrees of freedom in the problem by 1 by an appropriate choice of gauge. Thus  $z$  takes values on the manifold

$$\frac{U(N)}{U(N-1) \times U(1)},$$

which is better known as the complex projective space  $CP^{N-1}$ . Alternatively we can carry the gauge invariance along since it involves a compact gauge field and hence only contributes a finite multiplicative factor in finite volumes.

A similar limit yields the classical  $Sp(N)$  problem with energy

$$E = -\frac{1}{N} \sum_{\langle ij \rangle} |\mathcal{J}_{\alpha\beta} z_{i\alpha} z_{j\beta}|^2. \quad (5)$$

As advertised, in the following we will confine ourselves to the  $SU(N)$  problem.

## 2 Infinite $N$

### 2.1 General

If we now take the limit  $N \rightarrow \infty$ , the partition function can be evaluated using the standard saddle-point approximation. First, the constraint  $z_{i\alpha}^* z_{i\alpha} = N$  is enforced with the aid of a Lagrange multiplier  $\lambda_i$  on every site; the quartic interaction is made quadratic at the expense of introducing an auxiliary

complex variable  $Q_{ij}$  on every link:

$$Z = \int D\lambda DQ Dz e^{-\beta E[Q, \lambda, z]}, \quad (6)$$

where the effective energy is

$$\begin{aligned} E[Q, \lambda, z] &= - \sum_{\langle ij \rangle} (Q_{ij} z_{i\alpha}^* z_{j\alpha} + \text{C.c.} - N |Q_{ij}|^2) + i \sum_i \lambda_i (z_{i\alpha}^* z_{i\alpha} - N) \\ &= E[Q, \lambda, 0] + \sum_{i,j} z_{i\alpha}^* \mathcal{H}_{ij} z_{j\alpha}. \end{aligned} \quad (7)$$

Integration over the original variables  $z_{i\alpha}$  yields a new effective energy

$$E[Q, \lambda] = N \left( \sum_{\langle ij \rangle} |Q_{ij}|^2 - i \sum_i \lambda_i + \beta^{-1} \ln (\det \mathcal{H}[Q, \lambda]) \right). \quad (8)$$

In the limit  $N \rightarrow \infty$  the dominant contribution to the integral of  $e^{-\beta E[Q, \lambda]}$  comes from the vicinity of a saddle point,

$$\frac{\partial E[Q, \lambda]}{\partial Q_{ij}} = 0, \quad \frac{\partial E[Q, \lambda]}{\partial \lambda_i} = 0. \quad (9)$$

These conditions yield a set of self-consistent (mean-field) equations

$$NQ_{ij} = \langle z_{i\alpha} z_{j\alpha}^* \rangle, \quad N = \langle z_{i\alpha}^* z_{i\alpha} \rangle, \quad (10)$$

where  $\langle \dots \rangle$  denotes averaging over a thermal ensemble with energy

$$\sum_{i,j} \mathcal{H}_{ij} z_{i\alpha}^* z_{j\alpha} = \sum_i \mu_i z_{i\alpha}^* z_{i\alpha} - \sum_{\langle ij \rangle} (Q_{ij} z_{i\alpha}^* z_{j\alpha} + \text{C.c.}). \quad (11)$$

Thus  $Q_{ij} = Q_{ji}^*$  can be thought of as a hopping amplitude and  $\mu_i = i\lambda_i > 0$  as a chemical potential. Solving these equations requires either an explicit choice of gauge or a recognition that any non-trivial solution will really be a set of gauge equivalent solutions. Adding in all gauge equivalent saddle points would appear to restore manifest gauge invariance but an important subtlety in this procedure is the subject of Section 3.1.

Fluctuations of  $Q$  and  $\lambda$  contribute an amount of order  $1/N$  and can be neglected in the limit  $N \rightarrow \infty$ . In this approximation, free energy per boson

flavor is

$$\frac{F[Q, \mu]}{N} = \sum_{\langle ij \rangle} |Q_{ij}|^2 - \sum_i \mu_i + T \operatorname{Tr} \ln \mathcal{H}[Q, \mu]. \quad (12)$$

The last term — free energy of coupled harmonic oscillators  $z_i$  — involves the matrix  $\mathcal{H}$  with the following elements:

$$\mathcal{H}_{ij} = \begin{cases} \mu_i & \text{if } i = j, \\ -Q_{ij} & \text{if } i \text{ and } j \text{ are nearest neighbors,} \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

The saddle-point conditions (10) become

$$1 = \langle z_i z_i^* \rangle = T(\mathcal{H}^{-1})_{ii}, \quad Q_{ij} = \langle z_i z_j^* \rangle = T(\mathcal{H}^{-1})_{ij}. \quad (14)$$

(We have used the equipartition theorem for coupled harmonic oscillators  $z_i$ .)

In what follows we will explore states preserving time reversal symmetry; in such cases, one can choose a gauge where saddle-point values  $Q_{ij}$  are real. The set of equations (13) and (14) defines mean-field solutions of the large- $N$  model. This is a difficult nonlinear problem and an analytical solution is not always possible. However, there are a few helpful general results that we spell out below.

### 2.1.1 Solving for chemical potential

Any mean-field solution satisfies the equations

$$\mu_i = T + \sum_{j(i)} |Q_{ij}|^2, \quad (15)$$

where the sum is taken over nearest neighbors of site  $i$ . This result follows from the identity  $\sum_j (\mathcal{H}^{-1})_{ij} \mathcal{H}_{jk} = \delta_{ik}$  for  $i = k$ .

### 2.1.2 Trivial solution $Q = 0$

Mean-field equations (13) and (14) always have a trivial solution with  $Q_{ij} = 0$  on all links and  $\mu_i = T$ . ( $\mathcal{H}$  is proportional to the unit matrix:  $\mathcal{H} = T\mathbb{1}$ .) As we will see, this solution becomes a local minimum of free energy above

the temperature  $T_0 = 1$ . At high enough (but finite) temperatures, this is the *global* minimum.

### 2.1.3 Continuous phase transition at $T = 1$

At a high enough temperature, the system is in the random phase with  $Q_{ij} = 0$  everywhere. As temperature is lowered, we expect a transition into a phase where  $Q_{ij} \neq 0$  on some links. A continuous transition occurs when small fluctuations of  $Q_{ij}$  have a vanishing cost in terms of free energy. It therefore makes sense to expand free energy in powers of  $Q_{ij}$ :

$$\frac{F[Q, \mu] - F[0, T]}{N} = \sum_{\langle ij \rangle} |Q_{ij}|^2 - \sum_i (\mu_i - T) + T \operatorname{Tr} \ln \frac{\mathcal{H}[Q, \mu]}{T} \quad (16)$$

$$\begin{aligned} &= \operatorname{Tr} \left[ \frac{\mathcal{Q}^2}{2} - \mathcal{M} + T + T \ln \left( \frac{\mathcal{M} - \mathcal{Q}}{T} \right) \right] \\ &= \operatorname{Tr} \left[ \frac{\mathcal{Q}^2}{2} - T \sum_{n=2}^{\infty} \frac{1}{n} \left( \frac{\mathcal{Q} - \mathcal{M} + T}{T} \right)^n \right], \end{aligned} \quad (17)$$

where  $\mathcal{M}$  and  $-\mathcal{Q}$  are the diagonal and off-diagonal parts of matrix  $\mathcal{H} = \mathcal{M} - \mathcal{Q}$  (13). To eliminate the chemical potential  $\mathcal{M}$ , we use the constraint  $\langle z_i^* z_i \rangle = 1$ , or  $\partial F / \partial \mathcal{M}_{ii} = 0$ . By varying free energy (17) with respect to  $\mathcal{M}$ , we find that  $\mathcal{M} - T = \mathcal{O}(\mathcal{Q}^2)$ . Then, to order  $\mathcal{Q}^2$ ,

$$\frac{F[Q, \mu] - F[0, T]}{N} = \sum_{\langle ij \rangle} \frac{T-1}{T} |Q_{ij}|^2 + \dots \quad (18)$$

Eq. (18) shows that the random phase ( $Q_{ij} = 0$ ) becomes *locally* unstable for  $T < 1$ . Therefore, *if* the transition to a phase with  $Q_{ij} \neq 0$  is continuous, it must occur at the critical temperature  $T_c = 1$ .

It turns out, however, that in many cases the transition is *discontinuous* (see Table 1 in Section 2.3). Therefore, we should not truncate the expansion at order  $\mathcal{Q}^2$ : a cubic term or an expressly negative quartic one generally result in a first-order transition with  $T_c > 1$ . At intermediate temperatures  $1 < T < T_c$  the random phase  $Q = 0$  is locally stable but is not a global minimum.

What determines the order of the transition? Sokal and Starinets [5] suggest that it is the number of dimensions. They have shown that cubic lattices with  $d < 3/2$  exhibit a continuous transition, while those with  $d > 3/2$  have a discontinuous one. In the next few pages we will survey a few regular lattices with  $d = 0, 1, 2, 3$  and  $\infty$ . (The results are summarized at the beginning of Section 2.3 in Table 1.) It will be seen that the order of the transition, in fact,

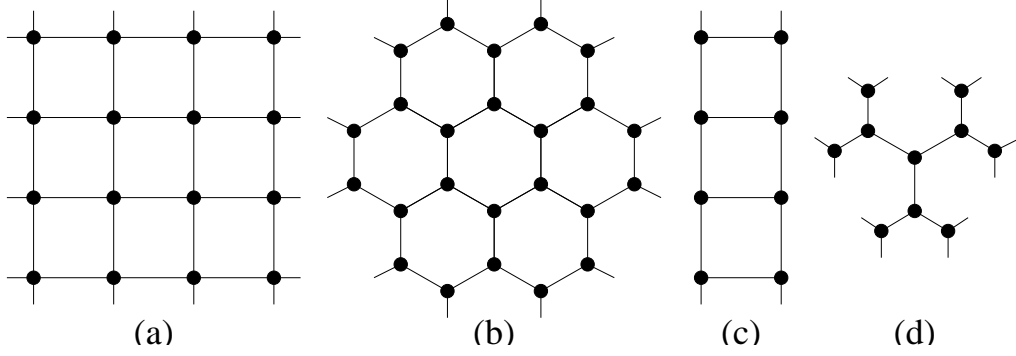


Fig. 1. Examples of regular lattices composed of equivalent sites: (a) square, (b) honeycomb, (c) ladder, (d) Bethe lattice.

has nothing to do with the number of dimensions! The relevant concept is, instead, the *local connectivity* of the lattice. Typically, the existence of short loops (of length 3 or 4) will make the transition discontinuous.

## 2.2 Regular lattices composed of equivalent sites

To make further progress, consider lattices composed of equivalent sites (Fig. 1). We will investigate solutions in which both the chemical potential and link variables are uniform, so that  $\mathcal{H} = \mu \mathbb{1} - Q\mathcal{A}$ , where  $\mathcal{A}$  is the adjacency matrix ( $\mathcal{A}_{ij} = A_{ji} = 1$  for nearest neighbors and 0 otherwise). Note that the link strength  $Q$  must be real in order for the matrix  $\mathcal{H}$  to be Hermitian.

The free energy now becomes a function of only two parameters  $Q$  and  $\mu$ :

$$\frac{F(Q, \mu) - F(0, T)}{NV} = \frac{zQ^2}{2} - \mu + T + \frac{T}{V} \sum_k \ln \left( \frac{\mu - Qa_k}{T} \right), \quad (19)$$

where  $a_k$  are eigenvalues of the adjacency matrix  $\mathcal{A}$  and  $z$  is the coordination number of the lattice. Variations with respect to  $\mu$  and  $Q$  yield the mean-field equations

$$1 = \frac{T}{V} \sum_k \frac{1}{\mu - Qa_k}, \quad (20)$$

$$zQ = \frac{T}{V} \sum_k \frac{a_k}{\mu - Qa_k} \quad (21)$$

One can easily derive from these an analogue of Eq. (15),

$$\mu - zQ^2 = T, \quad (22)$$

which can be used instead of Eq. (21).

### 2.2.1 $d = 0$ : two sites

The adjacency matrix has two eigenvalues,  $\pm 1$ , so that Eqs. (20-21) are easily solved to obtain an equation of state:

$$Q^2(Q^2 + T - 1) = 0. \quad (23)$$

At high temperatures,  $T > 1$ , there is only a trivial solution  $Q^2 = 0$ . Below  $T = 1$ , the global minimum of the free energy moves to  $Q^2 = 1 - T$  producing a continuous phase transition.

### 2.2.2 $d = 0, 1$ : periodic chain

For a chain of length  $L$  with periodic boundary conditions, the equation of state is obtained by solving Eqs. (20) and (22):

$$1 = \frac{1}{L} \sum_{n=1}^L \frac{T}{T - 2Q \cos(2\pi n/L) + 2Q^2}. \quad (24)$$

In several cases this can be done analytically:

$$L = 3 : \quad Q^2(T - 1 - Q + 2Q^2) = 0, \quad (25)$$

$$L = 4 : \quad Q^2[T - 1 + (T - 1 + 2Q^2)^2] = 0, \quad (26)$$

$$L = \infty : \quad Q^2(T - 1 + Q^2) = 0. \quad (27)$$

These are shown in Fig. 2.

The equation for a 3-site chain acquires nontrivial solutions when  $T \leq 9/8$ :

$$Q_1 = 0, \quad Q_2 = \frac{1 - \sqrt{9 - 8T}}{4}, \quad Q_3 = \frac{1 + \sqrt{9 - 8T}}{4}.$$

There is a first-order transition at  $T = T_c = 1.1107 \dots$ . At that temperature, the absolute minimum of the free energy  $F(Q, T)$  jumps from  $Q_1 = 0$  to  $Q_3 > 1/4$ , see Fig. 3.

At  $L = 4$ , the transition is continuous, although  $Q$  rises unusually steeply for a mean-field theory:  $Q \sim 2^{-1/2}(T_c - T)^{1/4}$ . For all  $L > 4$ , including the infinite chain, the transition is continuous with the usual exponent  $\beta = 1/2$ .

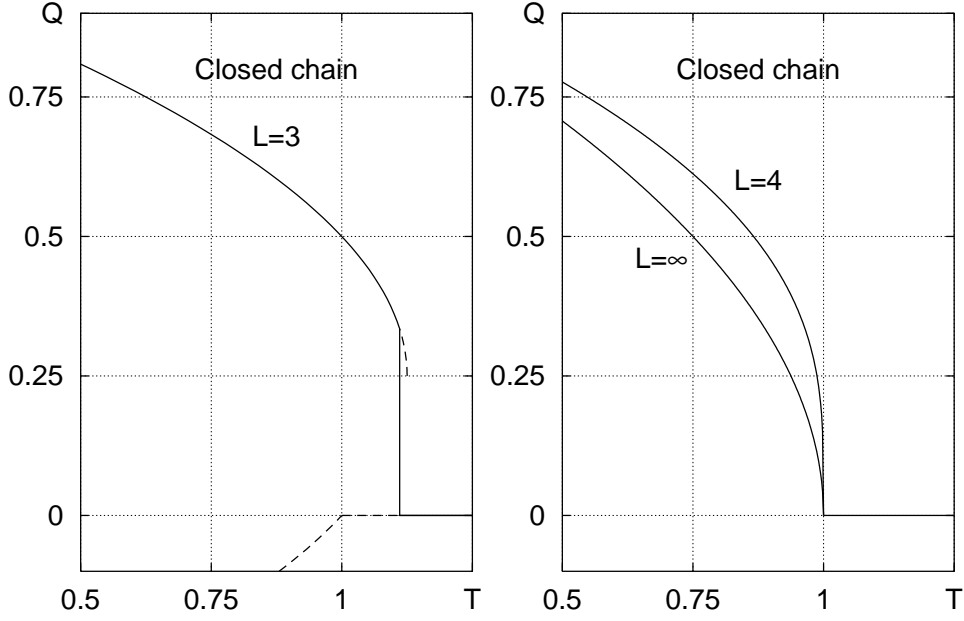


Fig. 2. Left: First-order phase transition on a closed chain of length  $L = 3$ . Dashed lines indicate metastable states. Right: second-order transition on closed chains of length  $L = 4$  and  $\infty$ .

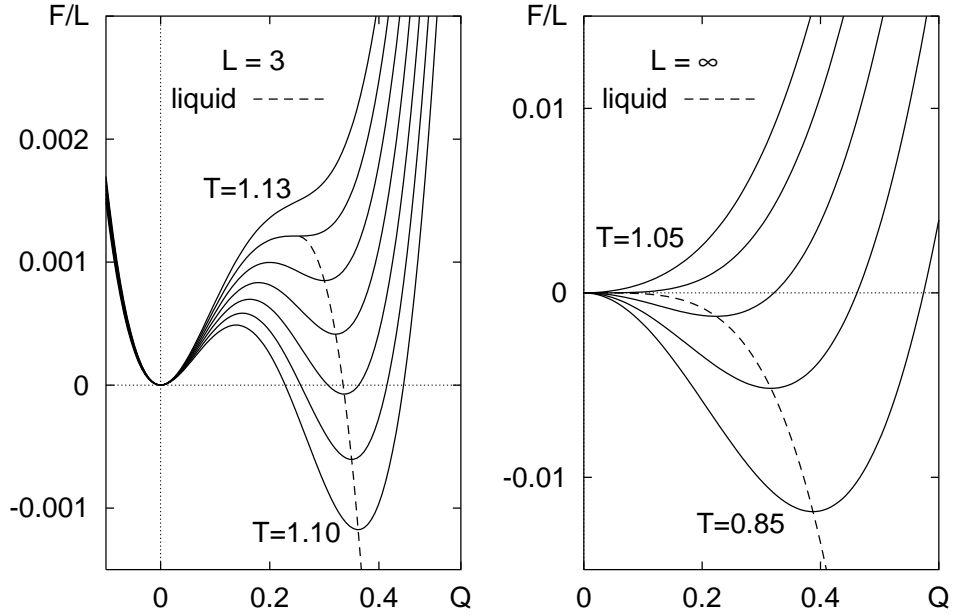


Fig. 3. Free energy per site  $F(Q, T)/L$ , periodic chains with  $L = 3$  (left) and  $L = \infty$  (right) for several temperatures around  $T_c$ . Dashed lines trace minima of the free energy corresponding to the liquid phase ( $Q \neq 0$ ). For  $L = 3$ , there is a first-order transition at  $T_c = 1.1107\dots$ . For  $L = \infty$ , the transition at  $T_c = 1$  is continuous.

### 2.2.3 $d = 2$ : square lattice

The equation of state on the infinite square lattice is again obtained by using the Eqs. (20) and (22):

$$1 = \int_0^{2\pi} \frac{dk_1}{2\pi} \int_0^{2\pi} \frac{dk_2}{2\pi} \frac{T}{T - 2Q(\cos k_1 + \cos k_2) + 4Q^2}. \quad (28)$$

There is only a trivial solution  $Q = 0$  at high temperatures,  $T \gg 1$ ; in the opposite limit,  $T \ll 1$ , there are two solutions,  $Q_1 = 0$  and  $Q_2 \approx 1$ , the latter being the global minimum of the free energy. The transition between the phases with  $Q = 0$  and  $Q \neq 0$  is discontinuous and takes place at  $T_c > 1$ . For if the phase transition were continuous, it would occur at  $T = 1$ , as we have argued previously;  $Q = 0$  would still be the global minimum of the free energy at the critical temperature. Instead, we will see that at  $T = 1$  the system is already in the phase with  $Q \neq 0$ , so that  $T_c > 1$ .

To see that Eq. (28) has more than one solution at  $T = 1$ , look at the behavior of its right-hand side. It diverges logarithmically at  $Q = 1/2$  and tends to 0 as  $Q \rightarrow \infty$ . By continuity, there must be a solution for  $Q > 1/2$  — in addition to the trivial one,  $Q = 0$ . Because at  $T = 1$  the system is already in the phase with  $Q \neq 0$ , the phase transition takes place at a higher temperature and is discontinuous.

The equation of state (28) can be integrated to obtain a closed form,

$$1 + \frac{4Q^2}{T} = \frac{2}{\pi} \mathbf{K} \left( \frac{4Q}{T + 4Q^2} \right), \quad (29)$$

where  $\mathbf{K}(k)$  is the complete elliptic integral of the first kind. The dependence  $Q(T)$  is shown in Fig. 4.

### 2.2.4 $d = 2$ : honeycomb lattice

One might think that the order of the transition is determined by the dimensionality of the lattice. This is not the case. While the transition on the two-dimensional square lattice is first-order, its counterpart on the honeycomb lattice is continuous (Fig. 4).

### 2.2.5 $d = 3$ : cubic lattice

This case has been discussed previously by Sokal and Starinets [5] for the  $SU(N)$  case and by DeSilva *et al.* [8] for  $Sp(N)$ . There is a first-order transition

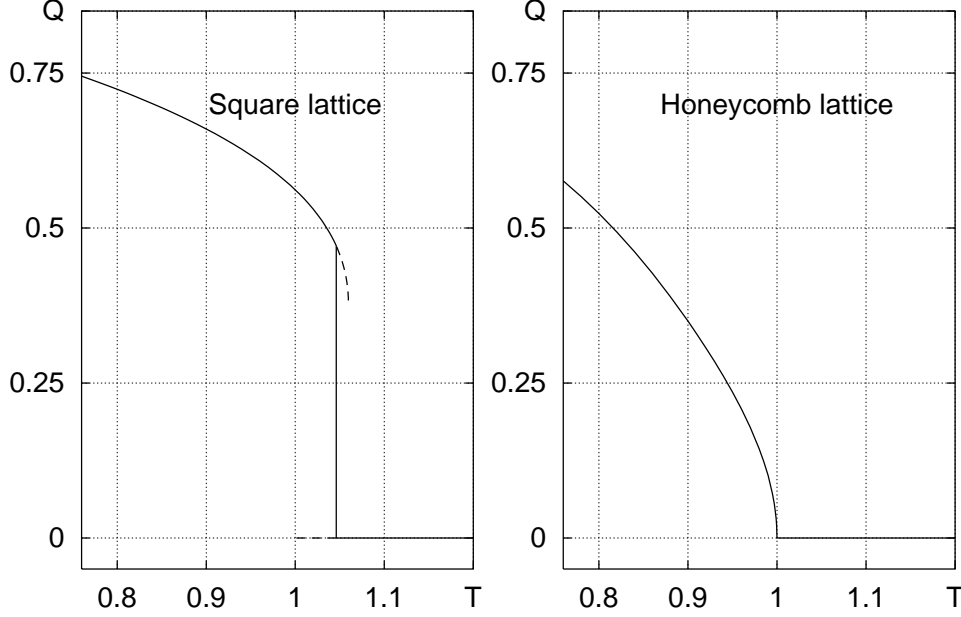


Fig. 4. Left: First-order phase transition on the square lattice. Dashed lines indicate metastable states. Right: second-order transition on the honeycomb lattice.

at  $T_c > 1$ . Unlike in lower dimensions, below  $T_c$  the system is in a ferromagnetic state that breaks the  $SU(N)$  symmetry. In an appropriate basis,

$$\langle z_{i1}^* z_{i1} \rangle \neq \langle z_{i2}^* z_{i2} \rangle = \langle z_{i3}^* z_{i3} \rangle = \dots = \langle z_{iN}^* z_{iN} \rangle.$$

It is interesting to note that the ferromagnetic transition in the  $d = 3$  Heisenberg model — the  $SU(2)$  case — is continuous. The change to a first-order transition at large  $N$  has a well-documented analogue in the two-dimensional Potts model [9]. There, the ferromagnetic transition is continuous for  $q \leq 4$  states and discontinuous for  $q > 4$ . We will see below why the Potts model is a simpler analog of our problem.

#### 2.2.6 $d = \infty$ : Bethe lattice

The Bethe lattice is another regular structure with equivalent sites (Fig. 1). When the number of nearest neighbors  $z \geq 3$ , the number of  $n$ -th neighbors grows faster than any power of  $n$ . In this sense, the Bethe lattice has an infinite number of dimensions.

The spectrum of the adjacency matrix for the Bethe lattice is well known [11]. Its eigenvalues fill the interval  $|a| < 2\sqrt{z-1}$  with density

$$\rho(a) = \frac{z\sqrt{4(z-1)-a^2}}{2\pi(z^2-a^2)}. \quad (30)$$

Table 1

Order of the  $N \rightarrow \infty$  phase transition and the exponent  $\beta$  for regular lattices.

lattice	$d$	order of transition	$\beta$	shortest cycle
chain, $L = 3$	0	1st	0	3
ladder	1	1st	0	4
triangular	2	1st	0	3
square	2	1st	0	4
cubic	3	1st	0	4
chain, $L = 4$	0	2nd	$1/4$	4
2 sites	0	2nd	$1/2$	none
chain, $L = \infty$	1	2nd	$1/2$	none
honeycomb	2	2nd	$1/2$	6
diamond	3	2nd?	$1/2?$	6
Bethe	$\infty$	2nd	$1/2$	none

Eq. (20) can now be integrated. The equation of state is

$$Q^2(Q^2 + T - 1) = 0. \quad (31)$$

Note that it is independent of the coordination number  $z$  and is therefore the same as that of an infinite chain (the Bethe lattice with  $z = 2$ ). The phase transition is continuous.

### 2.3 Expansion of the Landau free energy

Our survey of regular lattices is summarized in Table 1. It demonstrates that the order of the phase transition has nothing to do with the dimensionality of the system. This is not surprising: in the continuous version of the transition, the correlation length vanishes, instead of becoming infinite. Therefore, long-distance properties, such as dimensionality, are not important (in the limit  $N \rightarrow \infty$ ). Rather, details of the transition are determined by the local structure of the lattice. This connection is evident in Table 1: the transition is discontinuous when the lattice has closed loops of length 3 or 4. (The 4-site chain is a border case: although the transition is continuous, the exponent  $\beta = 1/4$  is unusually small.)

To substantiate this claim, we expand the free energy (19) in powers of  $Q$ :

$$\frac{F(Q, \mu) - F(0, T)}{NV} = \frac{zQ^2}{2} - \frac{T}{V} \sum_k \sum_{n=2}^{\infty} \frac{1}{n} \left( \frac{Qa_k - \mu + T}{T} \right)^n. \quad (32)$$

The term  $\mathcal{O}(Q^2)$  was written out in Eq. (18). To obtain an expansion to order  $\mathcal{O}(Q^4)$ ,  $\mu$  should be evaluated — to order  $Q^2$  — using the constraint  $\partial F/\partial \mu = 0$ , Eq. (20). We then obtain

$$\frac{F(Q, \mu) - F(0, T)}{NV} = \frac{T-1}{T} \frac{z_2 Q^2}{2} - \frac{z_3 Q^3}{3T^2} + \frac{(2z_2^2 - z_4) Q^4}{4T^3} + \mathcal{O}(Q^5). \quad (33)$$

The integers

$$z_n = \frac{1}{V} \sum_k a_k^n = \frac{1}{V} \text{Tr} \mathcal{A}^n$$

give the number of distinct loops of length  $n$  starting from some lattice node. In particular,  $z_2$  is simply the number of nearest neighbors  $z$ .

At high temperatures, the minimum of free energy (33) is at  $Q = 0$ . For  $T > 1$ , this point is a local minimum of  $F$ , at  $T < 1$  it becomes a local maximum. A necessary condition for a continuous phase transition at  $T_c = 1$  is the absence of a cubic term in this expansion [10]. A negative quartic term will also make the transition discontinuous.

### 2.3.1 Cubic term

A cubic term is only possible on lattices containing loops of length 3, in which case  $z_3 \neq 0$ . Therefore the transition is discontinuous in a closed chain of length 3 and on the triangular lattice.

### 2.3.2 Quartic term

The sign of the quartic term is determined by the presence of cycles of length 4. Generally, if there are no such loops, the quartic term is positive and the transition is likely continuous (barring a cubic term or some pathological behavior in higher orders). All lattices listed in the lower part of Table 1 look like trees if explored at depths up to 4. Continuous phase transitions are then possible.

The presence of loops of perimeter 4 makes the quartic term negative or — in special cases — zero. This alters the character of the transition and makes

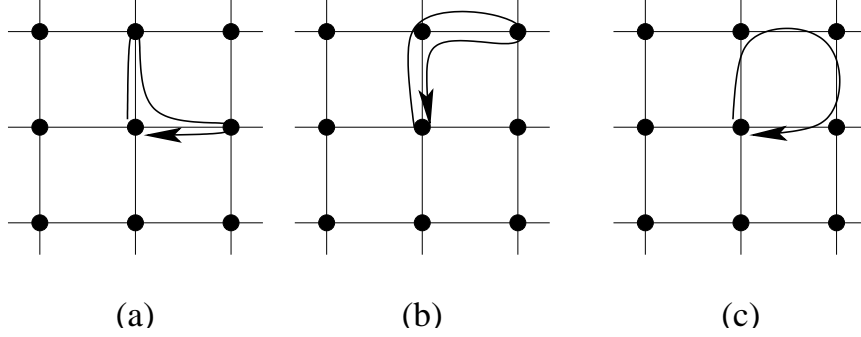


Fig. 5. There are  $z^2 + z(z - 1)$  noncyclic closed paths (a) and (b) of length 4 — in addition to cycles (c).

it first order. The remaining lattices in Table 1 (4-site chain, ladder, square, cubic) all have loops of length 4.

To prove these conjectures, let us evaluate the prefactor of the quartic term,  $2z_2^2 - z_4$ , that is responsible for its sign. On a tree without loops, the number of round-trip paths of length 4 is  $z^2 + z(z - 1)$ , see Fig. 5 (a,b). In this case,  $2z_2^2 - z_4 = z > 0$  and the quartic term is positive.

If, in addition,  $z_4$  contains  $l_4$  loops [Fig. 5 (c)], the quartic term is proportional to  $z - l_4$ . It is easy to see that the number of the loops  $l_4$  cannot be smaller than  $z$  (if they are present at all), hence the quartic term cannot be positive when such loops are present.

The quartic term is expressly negative for a ladder ( $z = 3$ ,  $l_4 = 4$ ), a square lattice ( $z = 4$ ,  $l_4 = 8$ ), and a cubic lattice ( $z = 6$ ,  $l_4 = 24$ ), hence first-order transitions. A chain of length 4 ( $z = 2$ ,  $l_4 = 2$ ) is a special case: the quartic term just vanishes. The transition is still continuous — thanks to a positive term  $\mathcal{O}(Q^6)$ ? — but  $Q$  rises more steeply than  $(T_c - T)^{1/2}$ , Fig. 2.

#### 2.4 *It's the connectivity, stupid!*

In this section, we have investigated a phase transition that occurs in an  $SU(N)$  ferromagnet at infinite  $N$ . It is a transition between a seriously disordered high- $T$  phase with no correlations between spins whatsoever and a low- $T$  phase that is either a paramagnet with short-range correlations (in low dimensions  $d \leq 3$ ), or possibly a ferromagnet.

The transition can be first or second-order, as noted by several authors previously [5,8]. Here we have uncovered what determines the order of the transition: it is the local connectivity of the lattice, rather than its dimensionality. In hindsight, this is not surprising because — in its continuous version — this is a transition between phases with finite and *zero* correlation lengths,

so it should be more sensitive to local features (presence of loops) than to long-distance ones (number of dimensions).

We have substantiated our claim by expanding the free energy in powers of an order parameter  $Q_{ij} = \langle z_i z_j^\dagger \rangle / N$  living on links. The presence of loops of length 3 leads to the existence of a cubic term in this expansion; loops of length 4 tend to make a quartic term negative. In either case, by the standard Landau argument, the transition becomes discontinuous. Finally a caveat on connectivity: our statements are for the purely nearest neighbor model. Interactions of further range will effectively change the connectivity.

### 3 Large $N$

So far we have discussed the limit of infinite  $N$ , when the saddle-point approximation is exact. We turn next to what happens when  $N$  is finite. One aspect is clear: the high-temperature phase is no longer perfectly disordered. Nevertheless, for sufficiently large  $N$ , the mean-field theory should be a good starting point for the analysis. To proceed further we will ask three questions. First, whether the  $N = \infty$  transition is characterized by an ordering that can also characterize a phase transition at finite  $N$ . Second, whether the finite  $N$  phase transition can be characterized by a different ordering that still continuously connects to the  $N = \infty$  transition. Third, we will examine the nature of the finite  $N$  corrections to see what they suggest about the fate of the transition.

We will consider these questions in turn, starting with the first question which has been discussed previously under the rubric of Elitzur's theorem for gauge theories.

#### 3.1 *Spontaneous breaking of a gauge symmetry at $N = \infty$*

As noted at the outset, our class of problems exhibit a local gauge invariance. In our  $N = \infty$  analysis we picked a gauge to identify a saddle point and left open the option of not picking a gauge and simply adding in all the gauge equivalent saddle points. While this would appear to be a manifestly gauge invariant way of proceeding, it hides an interesting anomaly noted most recently by Sokal and Starinets [5]: the local  $U(1)$  gauge symmetry appears to be spontaneously broken in the low-temperature phase if one invokes the usual criterion of a response to an infinitesimal field taken to zero post-thermodynamic

limit.<sup>3</sup> So this answers our first question: there *is* a broken symmetry at  $N = \infty$  but it cannot be broken at any finite  $N$  by virtue of Elitzur's theorem [12]. Sokal and Starinets go further and use this observation to conclude that “it seems unlikely that such a transition can survive to finite  $N$ .” We show in the following, by a simpler example, that this is a red herring and that at least in the case of a first order transition there is no contradiction between the restoration of Elitzur's theorem and the survival of the transition. The example involves the Potts model, which we will rewrite as an Ising gauge theory. This will enable us to appeal to well known results on the Potts model to establish the phase diagram of the theory. Altogether we will see the breakdown of gauge invariance is a consequence of something special—the divergence between the exchange constant and the transition temperature in the infinite  $N$  limit.

First let us note the violation of Elitzur's theorem explicitly. The gauge symmetry in question is evident from the form of energy (4). Changing the phase of the oscillator variable  $z_{i\alpha} \mapsto z_{i\alpha} e^{i\chi_i}$  leaves the energy invariant. However, the link variable  $Q_{ij} = \langle z_{i\alpha} z_{j\alpha}^* \rangle / N$  is certainly not invariant:  $Q_{ij} \mapsto Q_{ij} e^{i(\chi_i - \chi_j)}$ . Thus the absolute phases of  $Q_{ij}$  are immaterial. In particular, changing the sign of all link variables (on a bipartite lattice!) gives a physically equivalent configuration. Nevertheless, it can be shown that adding an infinitesimal gauge-fixing term to the energy,

$$E(\eta) = - \sum_{\langle ij \rangle} \left[ \frac{|z_{i\alpha}^* z_{j\alpha}|^2}{N} + \eta (z_{i\alpha}^* z_{j\alpha} + z_{j\alpha}^* z_{i\alpha}) \right], \quad (34)$$

leads to a non-analytic behavior of the free energy near  $\eta \rightarrow 0$ :

$$\frac{F[Q, \mu, \eta]}{N} = \frac{F[Q, \mu, 0]}{N} - 2|\eta| \sum_{\langle ij \rangle} |Q_{ij}| + \mathcal{O}(\eta^2). \quad (35)$$

(The limit  $N \rightarrow \infty$  should be taken first to ensure validity of the mean-field treatment.) The nonanalyticity of the free energy (35) means that the values of  $Q_{ij}$  are frozen either near  $+|Q_{ij}|$  or  $-|Q_{ij}|$ , depending on the sign of  $\eta$ . Thus the local U(1) gauge symmetry is spontaneously broken in a phase with  $|Q_{ij}| \neq 0$ .

### 3.1.1 Question

Why is a gauge symmetry spontaneously broken at large  $N$ ? What happens when the number of flavors is large but finite? To answer these questions, one

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<sup>3</sup> Spontaneous breaking of a gauge symmetry at large  $N$  has been investigated by several authors [13,14]

could study thermodynamics of an  $SU(N)$  ferromagnet in the presence of a small symmetry-breaking term (34). The partition function of an infinite chain (or just of a single link) can be evaluated for any finite  $N$  using the transfer matrix:

$$[Z(\beta, \eta)]^{1/L} = 2(N-1) \int_0^{\pi/2} \sin^{2N-3} \theta \cos \theta d\theta e^{N\beta(\cos^2 \theta - \eta \cos \theta)} \quad (36)$$

(see the companion paper [1] for details on the integration measure).

Evaluation of the partition function (36) is a feasible, though not particularly straightforward task. To keep technical details to a minimum, we have chosen to study a similar phenomenon in the Potts model. Because it involves discrete degrees of freedom, the broken gauge symmetry is also discrete ( $Z_2$ ).

### 3.1.2 Insights from the Potts model

We define the Potts model [9] in terms of unit vectors  $\mathbf{S}_i$  that can point along orthogonal axes in a  $q$ -dimensional internal space: the energy is

$$E = - \sum_{\langle ij \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)^2, \quad \mathbf{S}_i = \pm \hat{\mathbf{e}}_1, \pm \hat{\mathbf{e}}_2, \dots, \pm \hat{\mathbf{e}}_q, \quad \hat{\mathbf{e}}_m \cdot \hat{\mathbf{e}}_n = \delta_{mn}. \quad (37)$$

We have doubled the number of states per site compared to the usual amount. As a result, the energy is invariant under a local  $Z_2$  symmetry  $\mathbf{S}_i \mapsto -\mathbf{S}_i$ . Apart from a multiplicative factor, the partition function is identical to that of the Potts model. Parametrization (37) is a discrete analogue of representing  $SU(N)$  spins in terms of Schwinger bosons (4).

We can construct an infinite  $q$  treatment along the same lines as the large  $N$  treatment discussed previously: we decouple the quartic interaction in favour of a gauge field:

$$e^{\beta(\mathbf{S}_i \cdot \mathbf{S}_j)^2} = \sqrt{\beta/\pi} \int_{-\infty}^{\infty} dQ_{ij} e^{-\beta(Q_{ij}^2 - 2Q_{ij}\mathbf{S}_i \cdot \mathbf{S}_j)}. \quad (38)$$

and then integrate the spins out. The result of this analysis is the prediction of a finite temperature first order phase transition from a phase with zero correlation length as the temperature is lowered. Again the infinite  $q$  system breaks the Ising gauge invariance.

Now we know that for any finite  $q$ , the Potts model has a phase transition in  $d = 2$  or more but not in  $d = 1$  and that the transition is indeed first order at

large  $q$  in  $d \geq 2$  dimensions.<sup>4</sup> We also know that the finite  $q$  problem cannot break gauge invariance. Evidently the prediction of a first order transition in a dimension where it has every reason to be robust, is *not* vitiated by the breaking of local gauge invariance.

Indeed one can use this example to see where the breaking comes from. Consider for simplicity the one dimensional chain. The free energy density of an infinite chain can be evaluated with the aid of the transfer matrix yielding

$$f(\beta) = \lim_{L \rightarrow \infty} F(\beta)/L = -\beta^{-1} \ln(e^\beta + q - 1). \quad (39)$$

This is an analytic function of  $\beta$  for any finite  $q$ . At large  $q$ ,

$$f(\beta) \sim \begin{cases} -\beta^{-1} \ln q & \text{if } \beta < \ln q, \\ -1 & \text{if } \beta > \ln q. \end{cases} \quad (40)$$

In the limit  $q \rightarrow \infty$ , the free energy develops a kink at  $\beta = \ln q$ , so that there is a first-order phase transition.

The  $Z_2$  gauge symmetry is broken in the low-temperature phase ( $\beta > \ln q$ ). To see this, add a symmetry-breaking term  $-\eta \mathbf{S}_i \cdot \mathbf{S}_j$  to the energy of every bond and evaluate the free energy in its presence:

$$f(\beta, \eta) = -\beta^{-1} \ln(e^\beta \cosh \beta \eta + q - 1). \quad (41)$$

The expectation value of the gauge-dependent quantity

$$\langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle = -\frac{\partial f(\beta, \eta)}{\partial \eta} = \frac{\sinh \beta \eta}{\cosh \beta \eta + (q - 1)e^{-\beta}} \quad (42)$$

(the analogue of  $\langle z_i^\dagger z_i \rangle$ ) depends on the order of limits  $\eta \rightarrow 0$  and  $q \rightarrow \infty$ :

$$\langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle \sim \begin{cases} \beta \eta & \text{if } \eta \ll 1/\ln q \ll 1, \\ \text{sgn } \eta & \text{if } 1/\ln q \ll \eta \ll 1, \end{cases} \quad (43)$$

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<sup>4</sup> This treatment correctly predicts the order of the transition. However, the low-temperature phase is not a paramagnet: in  $d \geq 2$  dimensions, the discrete global symmetry of the Potts model can be spontaneously broken. In order to characterize a ferromagnetic phase, one must introduce an appropriate order parameter. This complication is absent in the  $SU(N)$  model in  $d = 2$  in view of the Mermin-Wagner theorem.

If the limit  $\eta \rightarrow 0$  is taken first (at finite  $q$ ), the gauge symmetry remains intact. Reversing the order of limits ( $q \rightarrow \infty$  first) leads to spontaneous breaking of the gauge symmetry.

### 3.1.3 Answer

What have we learned from the Potts model? Gauge symmetry can be spoiled by adding a term that prefers one gauge configuration over all others. The symmetry is spontaneously broken if gauge selection reliably occurs even when the gauge-fixing term is small. The example we have just worked out shows that there may be different degrees of smallness: the symmetry-breaking perturbation can be compared to the interaction strength, as well as to temperature. The presence of a large number of flavors in the model pushes the temperature scale down (as  $1/N$  in the Schwinger-boson case or as  $1/\ln q$  in the Potts model). Thus a nominally small gauge-fixing term ( $\eta \ll 1$ ) can still be large enough ( $T \ll \eta \ll 1$ ) to pick out a gauge. Most importantly, this does not rule out a persistence of the transition when  $q$  is finite, contradicting the belief of Sokal and Starinets.

## 3.2 Gauge-invariant order parameters?

The next order of business is to ask whether a gauge-invariant order parameter can discriminate between the putative finite  $N$  versions of the two phases. Note that both are  $SU(N)$  invariant. In our discussion of  $d \geq 3$  later we will briefly consider the new features that arise when that can be broken as well.

Away from infinite  $N$ ,  $\langle Q_{ij} \rangle = 0$  by gauge invariance while the amplitude  $\langle |Q_{ij}| \rangle$  will be non-zero in both phases although it could jump across the transition. At any rate it cannot serve as an order parameter. This leaves the gauge invariant fluxes that remain when the fluctuations of the amplitude are integrated out. In  $d = 1$  this sector is empty. In higher dimensions these would be described, at large  $N$ , by a weakly coupled  $U(1)$  gauge theory. While the coupling may jump across the phase boundary, at sufficiently large  $N$  it will be small on either side and we expect that the gauge fluctuations will be in the same phase on either side, i.e. barely confining in  $d \leq 3$  while deconfining in  $d \geq 4$ . In any event we do not expect a qualitative distinction to develop between the two phases and hence the transition between them will have the character of a liquid-gas transition, and not an order-disorder transition. For this reason, we expect the transition to be generically first order if it exists. A qualitative distinction *could* develop at smaller  $N$  in  $d \geq 3$  with a change in the character of the transition. We will briefly return to this later in our comments on  $d \geq 3$ .

An important comment is in order. From the perspective of the original quantum magnetic problem, we are operating at high temperatures and large “spins”. In this limit the temporal dimension has shrunk to zero and so we are computing Wilson loops which are not really diagnostic of the free energy cost of separating two spinons. The latter requires a computation of the temporal Polyakov loops and shows deconfinement consistent with the notion that the paramagnet is insensitive to the insertion or removal of a local fixed spin.

### 3.3 No phase transition in $d \leq 1$

Finite lattices and lattices in  $d = 1$  dimensions simply cannot have a phase transition when interactions are short ranged and  $N$  is finite. An infinite  $N$ , in essence, provides an extra dimension with a long-range interaction [every flavor interacts with every other one, see Eq. (1)] which allows this conclusion to be evaded. Thus in  $d \leq 1$  the transition will smooth into a crossover near  $T = 1$ . An explicit solution of the one-dimensional  $SU(N)$  and  $Sp(N)$  chains in the companion paper by Fendley and Tchernyshyov [1] confirms this; we direct the reader there for details of their method.

Here we content ourselves with following the smearing of the infinite- $N$  transition into a crossover in the case of a single link. The effective free energy in this case is

$$F(Q, i\lambda_1, i\lambda_2) = N[|Q|^2 - i(\lambda_1 + \lambda_2)N + \beta^{-1} \ln(-\lambda_1\lambda_2 - |Q|^2).] \quad (44)$$

As a function of  $\lambda_1$  and  $\lambda_2$ , it has a saddle point for real and positive

$$i\lambda_1 = i\lambda_2 = \mu(Q, T) = T/2 + \sqrt{(T/2)^2 + |Q|^2}.$$

Integrating out Gaussian fluctuations of  $\lambda$  gives a nonsingular contribution of order 1 to the free energy of the link:

$$F(Q) = N \left\{ |Q|^2 - 2\mu(Q, T) + T \ln [\beta\mu(Q, T)] \right\} + \mathcal{O}(1).$$

This free energy has a minimum at  $Q = 0$  if  $T > 1$ ; otherwise, the minimum is at  $|Q|^2 = 1 - T + \mathcal{O}(1/N)$ . For large  $N$  and  $T$  near 1, the partition function evaluates to

$$Z(\beta) = \int dQ^* dQ e^{-\beta F(Q)} \sim e^{N(1+\beta-\ln \beta)} \int_{T-1}^{\infty} dx e^{-Nx^2/2}.$$

As temperature crosses 1, energy of the link smoothly changes from being  $\mathcal{O}(N)$  to being  $\mathcal{O}(1)$ :

$$E(T) = -\frac{d \ln Z(\beta)}{d\beta} \sim \begin{cases} N(T-1) & \text{if } T < 1 \text{ and } 1/\sqrt{N} \ll |T-1| \ll 1, \\ -\sqrt{\pi N/2} & \text{if } |T-1| \ll 1/\sqrt{N}, \\ 1/(1-T) & \text{if } T > 1 \text{ and } 1/\sqrt{N} \ll |T-1| \ll 1. \end{cases}$$

The width of the crossover is  $\mathcal{O}(N^{-1/2})$ . As  $N \rightarrow \infty$ , the crossover turns into a phase transition.

### 3.4 Liquid-gas transition in $d = 2$

This is the most interesting case for our purposes. For one thing, the application of large  $N$  methods has been most influential with respect to ground state properties in  $d = 2$ . For another, the finite temperature problem cannot break the  $SU(N)$  symmetry in  $d = 2$  by Mermin-Wagner and so the intra-paramagnetic liquid-gas transition we have been considering is the only possibility for a thermal transition.

#### 3.4.1 First-order cases

In these cases, e.g. the square and triangular lattices, the transition should survive at sufficiently large  $N$ . Briefly, the infinite  $N$  transition involves the crossing of two separated saddle points. At large  $N$  we expect corrections to the contributions from the saddle points which are subdominant in  $N$  but these should not affect the existence of a crossing. The only way this can go wrong is if a whole host of other saddle points enter the finite  $N$  computation and their entropy overcomes the energy cost. This is what happens in  $d = 1$  where the other saddle points are domain walls (instantons) that turn the first order transition into a cross-over. (One can produce a first order transition in the infinite- $N$  problem in  $d = 1$  by adding a second neighbor coupling.)

In  $d = 2$ , as long as there is a finite surface tension per flavor between the two phases at  $N = \infty$ , any domain walls of order the system size should be exponentially suppressed and hence the phase transition should survive at sufficiently large  $N$ . As one of the two phases has zero correlation length, we have carried out this computation to see if the magnitude of the surface tension is anomalously small.

The computation is carried out by solving numerically the set of mean-field equations (13) and (14) on an  $L \times L$  square or triangular lattice at the co-

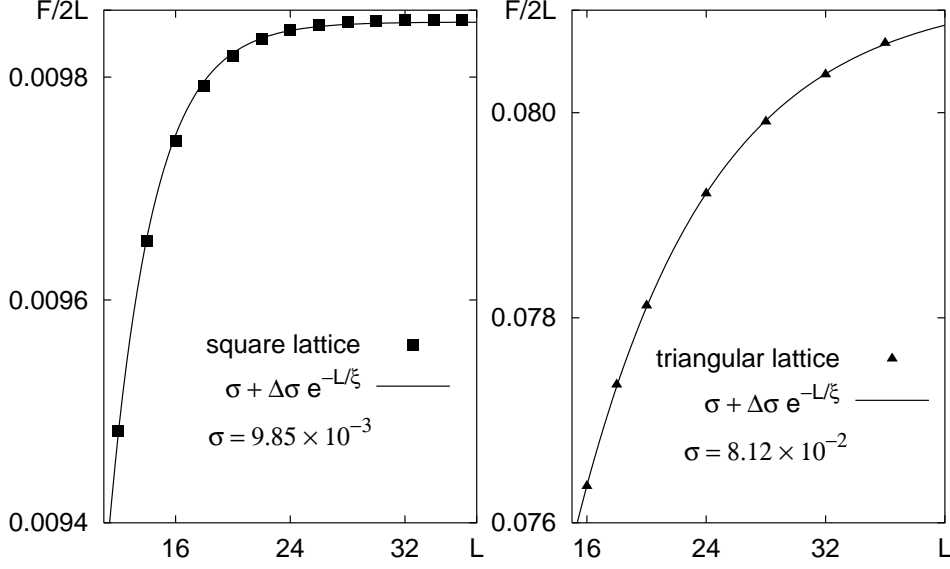


Fig. 6. Free energy per unit length of a domain wall on square and triangular lattices at their transition temperatures. Lines are fits to  $F/2L = \sigma + \Delta\sigma e^{-L/\xi}$ , where  $\sigma$  is the interface tension for an infinite domain wall,  $L$  is the system size, and  $\xi$  is the correlation length of Schwinger bosons in the liquid phase.

existence temperature of an infinite system. We have used periodic boundary conditions. A starting configuration includes two domains (liquid and gas) separated by two domain walls of total length  $2L$ . The free energy per unit length of the domain wall is fit to a simple form

$$F/2L = \sigma + \Delta\sigma e^{-L/\xi}.$$

The second term, a finite-size effect, contains the spinon correlation length  $\xi$  determined in the uniform liquid state leaving only two fitting parameters, interface tension  $\sigma$  and  $\Delta\sigma$  (Fig. 6).

For the square lattice this procedure yields a surface tension 0.01 per flavor per lattice constant and an interface of width  $\xi \approx 3$  lattice constants. For the triangular lattice this yields a surface tension 0.08 per flavor per lattice constant and an interface of width  $\xi \approx 9$  lattice constants. The surface tension is higher on a triangular lattice as expected since the first order transition is stronger in this case at  $N = \infty$ .

As the surface tension is finite, we conclude that immediately away from  $N = \infty$  both phases will get dressed by local fluctuations, whose detailed theory at large  $N$  is beyond our means at this point, but the transition will survive. We expect that the resulting line of first order transitions in the  $N, T$  plane will terminate, as in the liquid-gas problem, in a critical end point governed by the

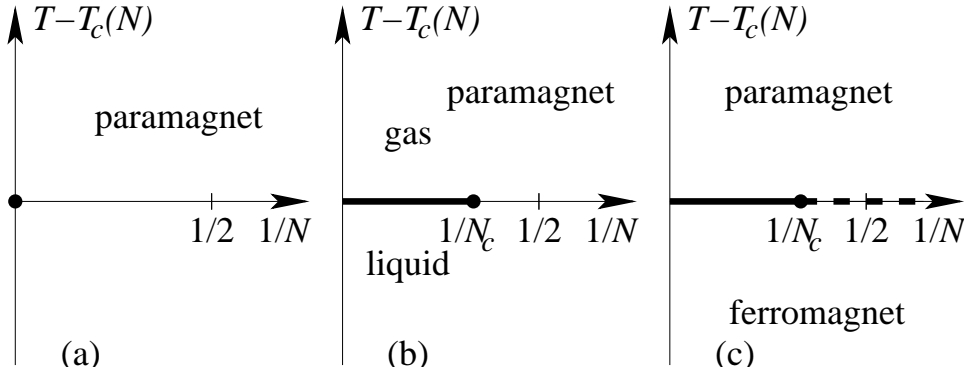


Fig. 7. Suggested phase diagrams of  $SU(N)$  ferromagnets. First and second-order transitions are shown as solid and dashed lines, respectively.

$d = 2$  Ising critical theory.<sup>5</sup> Of course, the location of this critical end point will be non-universal and is unlikely to lie at an integer value of  $N$ . In all of this, in a rough sense, valid near  $N = \infty$ , the temperature (or, rather,  $T - T_c$ ) plays the role of the field that discriminates between the two phases, by favoring the gas or liquid saddle point; while  $N$  itself is an overall multiplicative factor in the free energy (16), and therefore it acts as the inverse temperature. Indeed, infinite  $N$  corresponds to zero temperature in suppressing all fluctuations and making phase transitions possible even for finite lattices.

As the computed dimensionless surface tension for the square lattice comes out substantially smaller than 1, we suspect that the critical end point is not too far off and that one would need to go to very large  $N$  to see the first order transition. A precise estimate would require a theory of the fluctuations, which we do not have in hand at this point. Our computation on the triangular lattice suggests that one may be better off trying there. Of course in neither case does the phase transition survive to  $N = 2$  where one knows from studies of the Heisenberg model that it isn't there. (We believe that a similar estimate for the related  $RP^{N-1}$  problem will explain the failure of Sokal and collaborators to observe a first order transition in simulations for  $N$  as large as 8.)

### 3.4.2 Second-order cases

In these cases we conclude that the transition exists *only* at  $N = \infty$ . Essentially, this situation is the limiting case of the one in the previous section where the magnitude of the first order jump at  $N = \infty$  has gone to zero and hence the critical end point has been displaced all the way to  $N = \infty$ . The important caveat is that this limit is singular as far as the properties of the critical point are concerned. The “critical point” *at*  $N = \infty$  has no fluctuations and has zero correlation length. At large  $N$  there is a “boundary layer”

<sup>5</sup> Similarly to density in the liquid-gas problem, spin correlations  $|Q_{ij}|^2$  are discontinuous across the transition and thus play the role of an “order parameter”.

where the correlation length interpolates between a small microscopic value and infinity.

### 3.4.3 Transitions in $d \geq 3$

Starting with  $d = 3$  it becomes possible to break the  $SU(N)$  symmetry. A cursory examination of the  $N = \infty$  theory shows that cases in which the comparison of the paramagnetic solutions suggests a large jump in  $|Q_{ij}|$  end up going directly between the seriously disordered paramagnet and the “ferromagnet” in which  $SU(N)$  is broken. Consequently the transition survives when  $N$  is reduced and one expects that this line of first order transitions turns into a line of continuous transitions by the time the known  $SU(2)$  cases are reached.

The other new possibility, in  $d \geq 4$  is that the inter-paramagnetic transition survives for a range of  $N$  and then terminates in a line of confinement/deconfinement transitions. Deciding whether this is likely is beyond the methods used in this paper and of academic interest in the study of quantum magnets.

## 4 Conclusion

In the foregoing analysis we have established that the infinite  $N$  transition between two paramagnetic phases is *not* generically, an artefact of that limit. It *is* however a delicate transition, since it relies on a large entropy from the number of flavors overpowering the energetics. Consequently, as illustrated by our surface tension computation, we do not expect it to survive to small values of  $N$ . As such, while there is no contradiction between its existence at large  $N$  and the failure to observe it at values of  $N$  accesible by other methods, it does mean that the large  $N$  finite temperature phase diagram is not a reliable guide to the  $SU(2)$  case—which assumption is the basis of large- $N$  treatments. The same is true in higher dimensional cases where the infinite  $N$  transition is to the ferromagnetic phase. We have also shown that the violation of Elitzur’s theorem at infinite  $N$  is a consequence of a divergence between the exchange constant and the transition temperature and does not, in itself, invalidate the infinite  $N$  analysis.

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